WAVE PROPAGATION IN A VISCOUS FLUID CONTAINED IN AN ORTHOTROPIC ELASTIC TUBE

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ABSTRACT The problem of pressure wave propagation through a viscous fluid contained in an orthotropic elastic tube is considered in connection with arterial blood flow. Solutions to the fluid flow and elasticity equations are obtained for the presence of a reflected wave. Numerical results are presented for both isotropic and orthotropic elastic tubes. In particular, the pressure pulse, flow rate, axial fluid velocity, and wall displacements are plotted vs. time at various stations along the ascending aorta of man. The results indicate an increase in the peak value of the pressure pulse and a decrease in the flow rate as the pulse propagates away from the heart. Finally, the velocity of wave propagation depends mainly on the tangential modulus of elasticity of the arterial wall, and anisotropy of the wall accounts in part for the reduction of longitudinal movements and an increase in the hydraulic resistance.

INTRODUCTION

The problem of oscillatory flow of a viscous fluid contained in an elastic tube has a long history and continues to be of interest to the physiologists concerned with blood circulation. For a historical treatment of the entire subject one is referred to a paper by Lambossy (1951) and for more recent developments, the bibliographies contained in the books by McDonald (1960) and E. O. Attinger (1964).

Significant investigations of oscillatory flow of a viscous liquid in an elastic tube have been carried out by Morgan and Kiely (1954), Womersley (1957), and more recently by Whirlow and Rouleau (1965) and Atabek and Lew (1966). These investigators with the exception of Whirlow and Rouleau have assumed that the arterial wall may be approximated by a thin isotropic elastic tube. In reality the artery may have thick walls with viscoelastic or possibly anisotropic properties as indicated by Lawton (1955), Fenn (1957), Bergel (1961), and more recently by F. M. L. Attinger (1964) in their experimental studies.

Lambossy and Müller (1954) conducted a static analysis for the deformation of a closed anisotropic tube subjected to an internal pressure. Their analysis, however, was incorrect and Faucett (1957) in an attempt to correct this analysis also made the same fatal errors by not accounting for Maxwell's basic reciprocal relations of elasticity. Faucett's results however can readily be reduced to the correct results obtained recently by Mirsky (1967) in the studies of pulse velocities in an orthotropic elastic tube.

Although most of the mathematical models based on a dynamic analysis agree in part qualitatively with experimental findings, they still predict longitudinal wall movements far in excess of those ever observed experimentally. Womersley (1957) made an attempt to correct for this excess movement by the inclusion of an elastic constraint term which he supposed was due to a tethering effect. Studies by Patel and Fry (1965) reveal that this simple elastic tethering model may not be an adequate one. The possibility still exists that the arterial wall may behave elastically in an anisotropic manner and this behavior could account in part for the small longitudinal motions observed experimentally.

The present paper is an attempt to modify the Womersley model for arterial blood flow to account for the anisotropic behavior of the arterial wall. The approximate equations of motion for the arterial wall derived by Mirsky (1964) are applicable for thick-walled tubes and include isotropy as a special case. Of particular interest in the present studies are the flow rate, pressure, fluid velocities, and wall displacements. These parameters are computed for the ascending aorta of man from data obtained by Patel and Fry et al. (1965).

BASIC EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

In the present analysis we consider the motion of a viscous fluid in an orthotropic, thick elastic tube when subjected to a pressure gradient which is a periodic function of the time. It is assumed that small disturbances take place, hence the linearized form of Navier-Stokes equations is employed. With specified boundary conditions, these fluid equations of motion are solved in association with approximate equations of motion for an orthotropic elastic medium.

Equations of Motion for the Fluid

For a viscous incompressible fluid the equations of motion are governed by (*a*) the force equations expressing the conservation of momentum and (*b*) the continuity equation expressing the conservation of mass. For axisymmetric motions with respect to a cylindrical coordinate system (r, θ, x), these equations reduce to

$$\rho_f \frac{\partial v_r}{\partial t} = -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{1}{r^2} v_r + \frac{\partial^2 v_r}{\partial x^2} \right) \quad (1)$$

$$\rho_f \frac{\partial v_x}{\partial t} = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r} \frac{\partial v_x}{\partial r} + \frac{\partial^2 v_x}{\partial x^2} \right) \quad (2)$$

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} v_r + \frac{\partial v_x}{\partial x} = 0 \quad (3)$$

where ρ_f = density of the fluid medium, v_r, v_x = velocity components in the radial and axial directions respectively, P = total pressure, μ = viscosity of the fluid medium, and t = time.

In the development of these equations it has been further assumed that (*a*) blood viscosity is independent of the shear rate, (*b*) only laminar flow exists, i.e. no turbulence, and (*c*) arteries are sufficiently long and straight, hence the absence of tangential velocities.

Taylor (1959) investigated theoretically the influence of shear-dependent viscosity on oscillatory flow and also the effect of the presence of a thin marginal layer of plasma having a lower viscosity. He concluded that for the larger arteries only relatively small errors are introduced by the assumption of a constant viscosity coefficient. Assumptions (*b*) and (*c*) are open to question although there appears to be sufficient evidence to indicate that these effects are small.

Equations of Motion for the Arterial Wall

Mirsky (1964) developed an approximate set of equations for the dynamical behavior of thick cylindrical vessels possessing elastic properties which vary in the three mutually perpendicular directions. A material such as this is termed orthotropic and requires nine elastic constants for its specification. In the present analysis, these equations of motion are modified to include the effects of longitudinal tethering of the arterial wall and the viscosity of the fluid medium.

For axisymmetric motions, the components of the displacement in the axial and radial directions are approximated by

$$\bar{u}_x = u(x, t) + z\psi_x(x, t) \qquad (4)$$
$$\bar{u}_z = w(x, t) + z\psi_z(x, t)$$

where u, w are observed to be displacements of a particle on the middle surface $z = 0$, ψ_x is the angle of rotation of a normal to the middle surface in the $x - z$ plane, and ψ_z is the transverse normal strain. The coordinate z, taken to be positive outward in the direction of the normal to the middle surface, is integrated out in the development of the equations.

With the displacements specified by (4), the equations of motion are determined to be:

$$c_{33}\frac{\partial^2 u}{\partial x^2} + (h^2/12R)c_{33}\frac{\partial^2 \psi_x}{\partial x^2} + (c_{23}/R)\frac{\partial w}{\partial x} + c_{13}\frac{\partial \psi_z}{\partial x} + F_x/h$$
$$= \rho\left[\frac{\partial^2 u}{\partial t^2} + (h^2/12R)\frac{\partial^2 \psi_x}{\partial t^2}\right] + K[u + (h^2/12R)\psi_x] \qquad (5)$$

$$(h^3/12R)c_{33}\left(\frac{\partial^2 u}{\partial x^2} + R\frac{\partial^2 \psi_x}{\partial x^2}\right) + (h^3/12R)(c_{13} + c_{23})\frac{\partial \psi_z}{\partial x}$$
$$- \kappa_x^2 c_{44} h\left[\psi_x + \frac{\partial w}{\partial x} + (h^2/12R)\frac{\partial \psi_z}{\partial x}\right]$$
$$= \rho(h^3/12R)\left[R\frac{\partial^2 \psi_x}{\partial t^2} + \frac{\partial^2 u}{\partial t^2}\right] + K(h^3/12R)(u + R\psi_x) \qquad (6)$$

$$-(c_{23}/R)\frac{\partial u}{\partial x}+\kappa_z^2c_{44}\left[\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}+(h^2/12R)\frac{\partial^2\psi_z}{\partial x^2}\right]-(\bar{\alpha}c_{22}/Rh)w$$

$$-(1/Rh)(\bar{\beta}c_{22}+c_{12}h)\psi_z+q/h=\rho\left[\frac{\partial^2 w}{\partial t^2}+(h^2/12R)\frac{\partial^2\psi_z}{\partial t^2}\right] \quad (7)$$

$$-c_{13}h\frac{\partial u}{\partial x}+\kappa_z^2c_{44}(h^3/12R)\left[\frac{\partial\psi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}+R\frac{\partial^2\psi_z}{\partial x^2}\right]$$

$$-(h^3/12R)(c_{13}+c_{23})\frac{\partial\psi_x}{\partial x}-(c_{12}h+\bar{\beta}c_{22})\frac{w}{R}-\left(c_{11}h+\frac{\bar{\eta}}{R}c_{22}\right)\psi_z$$

$$=(\rho h^3/12)\left[\frac{\partial^2\psi_z}{\partial t^2}+\frac{1}{R}\frac{\partial^2 w}{\partial t^2}\right] \quad (8)$$

where c_{ij} = elastic constants, h = wall thickness, a, b = internal and external radii of vessel at mean pressure, R = mean radius of vessel = $\frac{1}{2}(a+b)$, κ_z^2 = shear coefficient (taken here as $\pi^2/12$), ρ = density of the elastic medium, $\bar{\alpha}=(h/R)[1+(h^2/12R^2)]$, $\bar{\beta}=-h^3/12R^2$, $\bar{\eta}=h^3/12R$, f_x, f_z = axial and radial components respectively of the external force/unit area acting at the boundary of the vessel walls,

$$(F_x,q)=\left[(f_x,f_z)\left(1+\frac{z}{R}\right)\right]_{z=-h/2}^{z=h/2},$$

and K = spring constant associated with the longitudinal tethering of the arterial wall.

It is of interest to note that for isotropic materials

$$c_{11}=c_{22}=c_{33}=E(1-\nu)/(1+\nu)(1-2\nu)$$

$$c_{12}=c_{13}=c_{23}=E\nu/(1+\nu)(1-2\nu) \quad (9)$$

where E, ν are respectively the Young's modulus and Poisson's ratio. Hence the special case $\nu=\frac{1}{2}$ must be excluded from the present analysis. This is not surprising since $\nu=\frac{1}{2}$ implies incompressibility and zero dilatation which is not accounted for in the approximate theory. Strictly speaking, the equations employed by Morgan and Kiely (1954) and Womersley (1955) are also not valid for $\nu=\frac{1}{2}$ since c_{11} has been replaced by $E/(1-\nu^2)$ and c_{12} by $E\nu/(1-\nu^2)$. These transformations arise from the additional assumptions made in the development of the membrane equations of motion. A more detailed discussion of this question is to be found in the work of Herrmann and Mirsky (1956) and Mirsky and Herrmann (1958).

Boundary Conditions

The boundary conditions associated with the fluid flow and elasticity equations may be specified as follows: (*a*) finiteness of the fluid velocities at $r=0$, (*b*) fluid and wall velocities are equal on surface of inner wall,

$$\frac{\partial u}{\partial t} - \frac{h}{2}\frac{\partial \psi_x}{\partial t} = (v_x)_{r=a}$$

$$\frac{\partial w}{\partial t} - \frac{h}{2}\frac{\partial \psi_z}{\partial t} = (v_r)_{r=a}, \tag{10}$$

and (*c*) the radial and shear stresses on the outer wall are zero and those acting on the inner wall are due to the fluid stresses.

$$f_x = 0 = f_z \quad \text{on} \quad z = \frac{h}{2}$$

$$f_x = \mu\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right) \quad \text{on} \quad z = -\frac{h}{2}\ (\text{or} \quad r = a)$$

$$f_z = -\left(p - 2\mu\frac{\partial v_r}{\partial r}\right) \quad \text{on} \quad r = a.$$

Note that condition (*c*) is incorporated in the quantities F_x, q appearing in the elasticity equations (5) and (7) with the result

$$F_x = -\mu\left(1 - \frac{h}{2R}\right)\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right)_{r=a}$$

$$q = \left(1 - \frac{h}{2R}\right)\left(p - 2\mu\frac{\partial v_r}{\partial r}\right)_{r=a} \tag{11}$$

where $p = P - P_{\text{mean}}$ is the pulsatile pressure.

SOLUTIONS TO FLUID FLOW AND ELASTICITY EQUATIONS

For plane waves propagating in the positive x direction, the pressure p and velocities v_r, v_x may be assumed in the form

$$p(r, x, t) = p_1(r) \exp[i(\omega t - \gamma x)]$$

$$v_r(r, x, t) = u_1(r) \exp[i(\omega t - \gamma x)]$$

$$v_x(r, x, t) = w_1(r) \exp[i(\omega t - \gamma x)]. \tag{12}$$

Here ω denotes the angular frequency of the forced oscillation and $\gamma = \omega/c$ (c = complex velocity) is a complex constant whose real part is the wave number and imaginary part is a measure of the decay of the disturbance as it progresses down the tube.

The solutions to equations (1), (2), and (3) satisfying condition (*a*) of (10) are well-known (see Womersley, 1955) and may be expressed as

$$p(r, x, t) = A_1 J_0(i\gamma r) \exp [i(\omega t - \gamma x)]$$

$$v_r(r, x, t) = [A_1(\gamma/\rho_f\omega) J_1(i\gamma r) - i\gamma A_2 J_1(\beta r)] \exp [i(\omega t - \gamma x)]$$

$$v_x(r, x, t) = [A_1(\gamma/\rho_f\omega) J_0(i\gamma r) - A_2\beta J_0(\beta r)] \exp [i(\omega t - \gamma x)] \qquad (13)$$

where A_1, A_2 are arbitrary constants of integration, $J_0(x)$, $J_1(x)$ are Bessel functions of the first kind, and

$$\beta^2 = -\left(\frac{\omega^2}{c^2} + \frac{i\omega}{\mu}\rho_f\right). \qquad (14)$$

If the elastic displacements are also assumed in the form

$$u = A_3 \exp [i(\omega t - \gamma x)]$$

$$\psi_x = A_4 \exp [i(\omega t - \gamma x)]$$

$$w = A_5 \exp [i(\omega t - \gamma x)]$$

$$\psi_z = A_6 \exp [i(\omega t - \gamma x)], \qquad (15)$$

we obtain the following set of six homogeneous equations in the arbitrary constants A_i.

$$A_1(\gamma/\rho_f\omega) J_0(i\gamma a) - A_2\beta J_0(\beta a) - i\omega A_3 + A_4(i\omega h/2) = 0$$

$$A_1(\gamma/\rho_f\omega) J_1(i\gamma a) - A_2 i\gamma J_1(\beta a) - i\omega A_5 + A_6(i\omega h/2) = 0$$

$$A_1\left(1 - \frac{h}{2R}\right)(2i\mu\gamma^2/\rho_f\,\omega h) J_1(i\gamma a) + A_2\frac{\mu}{h}\left(1 - \frac{h}{2R}\right)(\gamma^2 - \beta^2) J_1(\beta a)$$
$$+ A_3(\rho\omega^2 - \gamma^2 c_{33} - K) + A_4(h^2/12R)(\rho\omega^2 - \gamma^2 c_{33} - K)$$
$$- A_5(i\gamma c_{23}/R) - A_6(i\gamma c_{13}) = 0.$$

$$A_3(h^3/12\,R)(\rho\omega^2 - \gamma^2 c_{33} - K) + A_4[(h^3/12)(\rho\omega^2 - \gamma^2 c_{33} - K) - \kappa_x^2 c_{44} h]$$
$$+ A_5(i\gamma h\kappa_x^2 c_{44}) + A_6 i\gamma(h^3/12\,R)(\kappa_x^2 c_{44} - c_{13} - c_{23}) = 0$$

$$A_1\left(1 - \frac{h}{2R}\right)[J_0(i\gamma a) - (2\mu i\gamma^2/\rho_f\,\omega) J_1'(i\gamma a)]/h$$
$$+ A_2\left(1 - \frac{h}{2R}\right)(2\mu i\beta\gamma/h) J_1'(\beta a) + A_3(i\gamma c_{23}/R)$$
$$- A_4(i\gamma\kappa_x^2 c_{44}) + A_5[\rho\omega^2 - \gamma^2\kappa_x^2 c_{44} - (\bar{\alpha} c_{22}/Rh)]$$
$$+ A_6[(h^2/12R)(\rho\omega^2 - \gamma^2\kappa_x^2 c_{44}) - (\bar{\beta} c_{22} + c_{12}\,h)/Rh] = 0$$

$$A_3(i\gamma h c_{13}) + A_4 i\gamma(h^3/12\,R)(c_{13} + c_{23} - \kappa_x^2 c_{44})$$
$$+ A_5[(h^3/12\,R)(\rho\omega^2 - \kappa_x^2 c_{44}\gamma^2) - (c_{12}h + \bar{\beta} c_{22})/R]$$
$$+ A_6[(h^3/12)(\rho\omega^2 - \kappa_x^2 c_{44}\gamma^2) - (c_{11}h + \bar{\eta} c_{22}/R)] = 0. \quad (16)$$

Note that the first two equations of this set are obtained from the boundary conditions (10 *b*) and the latter four equations by direct substitution of expressions (15) into the set (5), (6), (7), and (8).

Velocity of Wave Propagation

For a nontrivial solution to the set of equations (16) we require the determinant of the coefficients of the A_i to vanish. This determinantal equation involves Bessel functions with arguments depending on c, the complex velocity of wave propagation. The equation is therefore transcendental in nature and theoretically there are an infinite number of roots for c. Fortunately at the highest frequencies likely to be of interest in the pulse wave, the quantity $\frac{\omega^2 a^2}{c^2} \ll 1$ and the Bessel functions may be represented in polynomial form. From a numerical point of view this is more convenient since the determinantal equation may now be expressed as a polynomial function of the wave velocity c. A more detailed discussion of the pulse velocity evaluation is presented later on in the analysis.

Solutions in Presence of a Reflected Wave

The previous analysis assumed that only a forward wave was present. This assumption is certainly not valid for the arterial system. In the presence of a reflected pressure wave the pressure p is the algebraic sum of the forward and retrograde wave and is given by

$$p(r, x, t) = A_1 J_0(i\gamma r) \exp[i(\omega t - \gamma x)] + A_1' J_0(i\gamma r) \exp[i(\omega t + \gamma x)]. \quad (17)$$

The radial and axial flow velocities are therefore expressed in the form

$$v_r(r, x, t) = [A_1(\gamma/\rho_f\omega) J_1(i\gamma r) - i\gamma A_2 J_1(\beta r)] \exp[i(\omega t - \gamma x)]$$
$$+ [A_1'(\gamma/\rho_f\omega) J_1(i\gamma r) + i\gamma A_2' J_1(\beta r)] \exp[i(\omega t + \gamma x)]$$
$$v_x(r, x, t) = [A_1(\gamma/\rho_f\omega) J_0(i\gamma r) - A_2 \beta J_0(\beta r)] \exp[i(\omega t - \gamma x)]$$
$$- [A_1'(\gamma/\rho_f\omega) J_0(i\gamma r) + A_2'\beta J_0(\beta r)] \exp[i(\omega t + \gamma x)]. \quad (18)$$

It can readily be shown that

$$A_2'/A_1' = -A_2/A_1\,. \quad (19)$$

Hence the velocities may be written as

$$v_r(r, x, t) = [(\gamma/\rho_f\omega)J_1(i\gamma r) - i\gamma(A_2/A_1)J_1(\beta r)]$$
$$\times [A_1 \exp i(\omega t - \gamma x) + A_1' \exp i(\omega t + \gamma x)]$$
$$v_x(r, x, t) = [(\gamma/\rho_f\omega)J_0(i\gamma r) - \beta(A_2/A_1)J_0(\beta r)]$$
$$\times [A_1 \exp i(\omega t - \gamma x) - A_1' \exp i(\omega t + \gamma x)]. \tag{20}$$

A similar analysis yields the following expressions for the elastic displacements:

$$u = (A_3/A_1)[A_1 \exp i(\omega t - \gamma x) - A_1' \exp i(\omega t + \gamma x)]$$
$$\psi_x = (A_4/A_1)[A_1 \exp i(\omega t - \gamma x) - A_1' \exp i(\omega t + \gamma x)]$$
$$w = (A_5/A_1)[A_1 \exp i(\omega t - \gamma x) + A_1' \exp i(\omega t + \gamma x)]$$
$$\psi_z = (A_6/A_1)[A_1 \exp i(\omega t - \gamma x) + A_1' \exp i(\omega t + \gamma x)] \tag{21}$$

where use has been made of the relations

$$(A_3'/A_1') = -(A_3/A_1) \qquad (A_4'/A_1') = -(A_4/A_1)$$
$$(A_5'/A_1') = (A_5/A_1) \qquad (A_6'/A_1') = (A_6/A_1). \tag{22}$$

If now we consider average values for v_x and p defined by

$$\bar{v}_x(x, t) = \frac{1}{\pi a^2}\int_0^a 2\pi r v_x \, dr$$
$$= (2/a)[(1/\rho_f\, \omega i)J_1(i\gamma a) - (A_2/A_1)J_1(\beta a)]$$
$$\times [A_1 \exp i(\omega t - \gamma x) - A_1' \exp i(\omega t + \gamma x)]$$
$$\bar{p}(x, t) = \frac{1}{\pi a^2}\int_0^a 2\pi r p \, dr$$
$$= (2/ai\gamma)[A_1 \exp i(\omega t - \gamma x) + A_1' \exp i(\omega t + \gamma x)]J_1(i\gamma a), \tag{23}$$

the hydraulic impedance Z may be written as

$$Z = -(\partial\bar{p}/\partial x)/\bar{v}_x$$
$$= i\rho_f\,\omega \Big/ \left[1 - i\rho_f\,\omega(A_2/A_1)\,\frac{J_1(\beta a)}{J_1(i\gamma a)}\right]$$
$$= R + iL. \tag{24}$$

One might add that the expression (24) for the impedance remains unaltered if no reflected wave is present.

NUMERICAL EXAMPLES

Velocity of Wave Propagation

As stated previously, if $\omega a/c \ll 1$, the evaluation of the pulse velocities is simplified by performing the following approximations to the Bessel functions:

$$J_0(i\gamma a) \sim 1 + \tfrac{1}{4}\gamma^2 a^2$$
$$J_1(i\gamma a) \sim \tfrac{1}{2}i\gamma a(1 + \tfrac{1}{8}\gamma^2 a^2)$$
$$J_0(\beta a) \sim J_0(\delta) + (\gamma^2 a^2/2\delta) J_1(\delta)$$
$$J_1(\beta a) \sim J_1(\delta) + (\gamma^2 a^2/2\delta^2)[J_1(\delta) - \delta J_0(\delta)] \qquad (25)$$

where

$$\delta = \alpha i^{3/2}$$
$$\alpha^2 = n\omega a^2 \rho_f/\mu \qquad (n = 1, 2, 3 \cdots).$$

The integer n refers to the particular harmonic component in question.

In dimensionless form, therefore, the determinantal equation for the pulse velocities can be written as

$$\text{det. } (a_{ij}) = 0 \qquad (i, j = 1, \cdots 6) \qquad (26)$$

where

$$m = h/R$$
$$y = n\omega a/c$$
$$\bar{q} = h/a$$
$$\overline{K} = Ka^2/c_{44}$$
$$b_{ij} = c_{ij}/c_{44}$$
$$a_{11} = 1 + 0.25\, y^2$$
$$a_{12} = -\delta\left[J_0(\delta) - 0.5y^2\left\{\frac{1}{\delta^2} J_0(\delta) - \frac{1}{\delta} J_1(\delta)\right\}\right]$$
$$a_{13} = -i$$
$$a_{14} = 0.5\, i$$
$$a_{15} = a_{16} = 0$$
$$a_{21} = 0.5\, i(1 + 0.125\, y^2)$$

$$a_{22} = -i\left[J_1(\delta) + 0.5y^2\left\{\frac{1}{\delta^2}J_1(\delta) - \frac{1}{\delta}J_0(\delta)\right\}\right]$$

$$a_{23} = a_{24} = 0$$

$$a_{25} = -i$$

$$a_{26} = 0.5\,i$$

$$a_{31} = a_{32} = 0$$

$$a_{33} = \frac{1}{12}m\bar{q}^2\left(\frac{n^2\omega^2a^2\rho}{c_{44}} - \bar{K} - b_{33}\,y^2\right)$$

$$a_{34} = \frac{1}{m}\,a_{33} - \kappa_x^{\,2}$$

$$a_{35} = i\kappa_x^{\,2}\bar{q}y^2$$

$$a_{36} = -\tfrac{1}{12}im\bar{q}(b_{13} + b_{23} - \kappa_x^{\,2})y^2$$

$$a_{41} = a_{42} = 0$$

$$a_{43} = i\bar{q}b_{13}$$

$$a_{44} = \tfrac{1}{12}im\bar{q}(b_{13} + b_{23} - \kappa_x^{\,2})$$

$$a_{45} = \frac{1}{12}m\bar{q}\,\frac{n^2\omega^2a^2\rho}{c_{44}} - m\left(b_{12} - \frac{m^2}{12}\,b_{22}\right) - \frac{1}{12}m\bar{q}^2\kappa_x^{\,2}y$$

$$a_{46} = \frac{1}{12}\,\bar{q}^2\,\frac{n^2\omega^2a^2\rho}{c_{44}} - \left(b_{11} + \frac{1}{12}\,m^2b_{22}\right) - \frac{1}{12}\,\kappa_x^{\,2}\bar{q}^2y^2$$

$$a_{51} = -\left(1 - \frac{m}{2}\right)y^2(1 + 0.125y^2)$$

$$a_{52} = \left(1 - \frac{m}{2}\right)[-\delta^2J_1(\delta) + y^2\{1.5J_1(\delta) + 0.5\delta J_0(\delta)\}]$$

$$a_{53} = (c_{44}\bar{q}/n\mu\omega)[(n^2\omega^2a^2\rho/c_{44}) - \bar{K} - b_{33}y^2]$$

$$a_{54} = c_{44}a_{33}/n\mu\omega\bar{q}$$

$$a_{55} = -imc_{44}b_{23}y^2/\mu n\omega$$

$$a_{56} = -ic_{44}b_{13}y^2/\mu n\omega$$

$$a_{61} = \left(1 - \frac{m}{2}\right)[1 + 0.25y^2 - (y^2/\delta^2)]$$

$$a_{62} = (2y^2/\delta^2)\left(1 - \frac{m}{2}\right)[\delta J_0(\delta) - J_1(\delta) + (0.5y^2/\delta^2)(\delta^2 - 1)J_1(\delta)]$$

$$a_{63} = imb_{23}c_{44}y^2/\rho_f n^2\omega^2a^2$$

$$a_{64} = -i\kappa_x^{\,2}c_{44}y^2/\rho_f n^2\omega^2a^2$$

$$a_{65} = (c_{44}\,\bar{q}y^2/\rho_f\, n^2\omega^2 a^2)\left[\frac{n^2\omega^2 a^2 \rho}{c_{44}} - \frac{m^2 b_{22}}{\bar{q}^2}\left(1 + \frac{m^2}{12}\right) - \kappa_x^2 y^2\right]$$

$$a_{66} = c_{44} y^2 a_{45}/\bar{q}\rho_f n^2\omega^2 a^2. \tag{27}$$

The expansion of the determinant (26) results in a polynomial of the seventh degree in y^2. Hence the roots represent the velocities of outgoing and incoming waves. In the present analysis only the highest roots for y (corresponding to the lowest values for c) are considered since waves propagating at the higher velocities have not been observed experimentally. It should be noted that the pulse velocity is frequency dependent and is computed for each harmonic component of the pressure wave. In particular, if the complex wave velocity is given in the form

$$c = c_{1n} + ic_{2n}\,,$$

the phase velocity c_p is

$$c_p = (c_{1n}^2 + c_{2n}^2)/c_{1n}\,.$$

Evaluation of the Physical Parameters

The computation of the pressure, flow rate, and displacements depends on the evaluation of the arbitrary complex constants A_1 and A_1'. These constants are obtained by matching the harmonic components of average pressure and flow rate (at $x = 0$) with experimental data of pressure and flow rate. In complex form, the average pressure and flow rate are given by

$$\begin{aligned}\bar{p}(x, t) &= \frac{1}{\pi a^2}\int_0^a p(r, x, t)\cdot 2\pi r\, dr \\ &= \sum_{n=1}^{s} (2/ia\gamma_n)[A_{1n} \exp i(n\omega t - \gamma_n x) + A_{1n}' \exp i(n\omega t + \gamma_n x)]J_1(i\gamma_n a) \\ Q(x, t) &= \int_0^a 2\pi r v_x(r, x, t)\, dr \\ &= 2\pi a[(1/i\rho_f\,\omega n)J_1(i\gamma_n a) - (A_{2n}/A_{1n})J_1(\beta_n a)] \\ &\qquad \times [A_{1n} \exp i(n\omega t - \gamma_n x) - A_{1n}' \exp i(n\omega t + \gamma_n x)]\end{aligned} \tag{28}$$

where

$$\gamma_n = n\omega/c_n$$

$$\beta_n^2 = -\left(\frac{n^2\omega^2}{c_n^2} + \frac{in\omega}{\mu}\rho_f\right).$$

For $x = 0$, the real parts of the expressions (28) may be written as

$$P(0, t) = \sum_{n=1}^{s} P_n \sin (n\omega t + \theta_n)$$

$$Q(0, t) = \sum_{n=1}^{s} Q_n \sin (n\omega t + \varphi_n) \tag{29}$$

which is the form of the experimental data given by Patel and Fry et al. (1965) for the ascending aorta of man.

The axial and radial displacements on the inner surface of the tube are

$$\bar{u}_{xa} = \sum_{n=1}^{s} (U_{1n} - \tfrac{1}{2}hU_{2n})[A_{1n} \exp i(n\omega t - \gamma_n x) - A'_{1n} \exp i(n\omega t + \gamma_n x)]$$

$$\bar{u}_{za} = \sum_{n=1}^{s} (W_{1n} - \tfrac{1}{2}hW_{2n})[A_{1n} \exp i(n\omega t - \gamma_n x) + A'_{1n} \exp i(n\omega t + \gamma_n x)] \tag{30}$$

where

$$U_{1n} = A_{3n}/A_{1n} \qquad U_{2n} = A_{4n}/A_{1n}$$

$$W_{1n} = A_{5n}/A_{1n} \qquad W_{2n} = A_{6n}/A_{1n}$$

are evaluated from the homogeneous system of equations (16). A similar set of expressions may be given for v_r and v_x. Finally for comparison of data from animal to animal it is found convenient to nondimensionalize the impedance and write

$$R_{an} = (a^2/8\mu)R_n$$

$$L_{an} = L_n/\rho_f n\omega \tag{31}$$

where

$$Z_n = i\rho_f n\omega \Bigg/ \left[1 - i\rho_f n\omega(A_{2n}/A_{1n}) \frac{J_1(\beta_n a)}{J_1(i\gamma_n a)}\right] = R_n + iL_n. \tag{32}$$

Data

Although data is available for the various elastic moduli (see McDonald (1960) and Patel et al. (1964)), there is no one complete set of orthotropic elastic data which includes values for the various Poisson ratios. It has been brought to the attention of the author that Atabek and his associates are currently working with a group at National Institutes of Health to determine the mechanical properties of arteries. Until more realistic data becomes available one can only make educated guesses for the elastic constants. In the evaluation of the figures the following three sets of data have been employed which include the special case of isotropy.

$m = 0.143 \qquad \bar{q} = 0.154 \qquad \overline{K} = 0 \qquad \kappa_x^2 = 0.8225$

$s = 10 \qquad \omega = 7.854 \text{ rad/sec} \qquad h = 0.2 \text{ cm} \qquad a = 1.3 \text{ cm}$

$\mu = 0.04 \text{ g/cm-sec} \qquad \rho = \rho_f = 1.05 \text{ g/cm}^3$

$E = E_\theta = 4 \times 10^6 \text{ dynes/cm}^2 \qquad c_o = 541 \text{ cm/sec}$

Case 1. (*Isotropic*)

$$b_{11} = b_{22} = b_{33} = 51.0 \qquad \nu = 0.49$$

$$b_{12} = b_{13} = b_{23} = 49.0$$

Case 2. (*Orthotropic*)

$$b_{11} = 0.925 \qquad b_{22} = 3.36 \qquad b_{33} = 1.22$$

$$b_{12} = 0.496 \qquad b_{13} = 0.42 \qquad b_{23} = 0.534$$

These values correspond approximately to the case

$$E_r = \tfrac{1}{4}E_\theta \qquad E_x = \tfrac{1}{3}E_\theta \qquad c_{44} = \tfrac{1}{3}E_\theta$$

where E_r, E_θ, and E_x are the Young's moduli.

Case 3. (*Orthotropic*)

$$b_{11} = 0.5 \qquad b_{22} = 2.0 \qquad b_{33} = 0.3333 \qquad \nu_{ij} = 0$$

$$b_{12} = b_{13} = b_{23} = 0$$

$$E_r = \tfrac{1}{4}E_\theta \qquad E_x = \tfrac{1}{3}E_\theta \qquad c_{44} = \tfrac{1}{2}E_\theta$$

It should be noted that for case 2 the Poisson ratios ν_{ij} were given as

$$\nu_{x\theta} = \nu_{r\theta} = 0.1, \qquad \nu_{xr} = \nu_{\theta r} = 0.4, \qquad \nu_{rx} = \nu_{\theta x} = 0.3.$$

These values were chosen so as to be consistent with the Maxwell reciprocity relations

$$\nu_{r\theta}E_\theta = \nu_{\theta r}E_r, \qquad \nu_{rx}E_x = \nu_{xr}E_r, \qquad \nu_{\theta x}E_x = \nu_{x\theta}E_\theta$$

where E_r, E_θ, and E_x are respectively the elastic moduli in the radial, tangential, and axial directions. The Poisson ratios ν_{ij} taken to be zero in case 3 represent an extreme case and should not be interpreted as being the situation occurring in arteries.

TABLE I

PHASE VELOCITIES (CM/SEC)

n	c_{p1}			c_{p2}		
	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
1	510	499	502	1980	1020	716
2	513	503	507	2050	1050	738
3	515	505	510	2080	1070	748
4	515	505	511	2100	1080	754
5	516	506	512	2110	1080	758
6	516	506	512	2120	1090	761
7	516	506	512	2130	1090	764
8	516	506	513	2140	1090	766
9	516	506	513	2140	1100	768
10	516	506	513	2150	1100	769

TABLE II

HYDRAULIC RESISTANCE $R_{\alpha n}$ AND INDUCTANCE $L_{\alpha n}$

n	Case 1		Case 2		Case 3	
	$R_{\alpha n}$	$L_{\alpha n}$	$R_{\alpha n}$	$L_{\alpha n}$	$R_{\alpha n}$	$L_{\alpha n}$
1	2.86	1.06	3.43	1.06	4.99	1.07
2	3.88	1.04	4.55	1.04	6.29	1.05
3	4.67	1.03	5.41	1.04	7.32	1.04
4	5.34	1.03	6.14	1.03	8.18	1.04
5	5.93	1.03	6.79	1.03	8.95	1.03
6	6.46	1.02	7.38	1.03	9.65	1.03
7	6.96	1.02	7.92	1.02	10.3	1.03
8	7.42	1.02	8.42	1.02	10.9	1.03
9	7.85	1.02	8.90	1.02	11.5	1.03
10	8.27	1.02	9.35	1.02	12.0	1.02

Pressure and Flow Data from Ascending Aorta

HARMONICS

Curve	Const. term	1		2		3		4		5		6		7		8		9		10	
		M	ψ	M	ψ	M	ψ	M	ψ	M	ψ	M	ψ	M	ψ	M	ψ	M	ψ	M	ψ
P (cm H_2O)	94	14.0	311	4.9	258	2.4	212	0.8	212	1.5	197	1.4	96	0.5	325	0.8	140	0.8	5	0.3	214
Q (cm^3/sec)	67	120	9	88	289	45	208	8	161	18	185	17	99	5	347	6	109	8	356	4	223

DISCUSSION OF RESULTS AND CONCLUSIONS

The results of phase velocity vs. parameter n are given in Table I for an isotropic material (case 1) and two orthotropic materials (cases 2 and 3). For the isotropic material these results agree closely with those of Womersley and others. c_{p1} and

c_{p2} are the propagation velocities of the first two outgoing waves which have been described previously by Atabek and Lew (1966). Note that for large values of n, i.e. frequency, the velocity of propagation approaches a constant value and for case 1

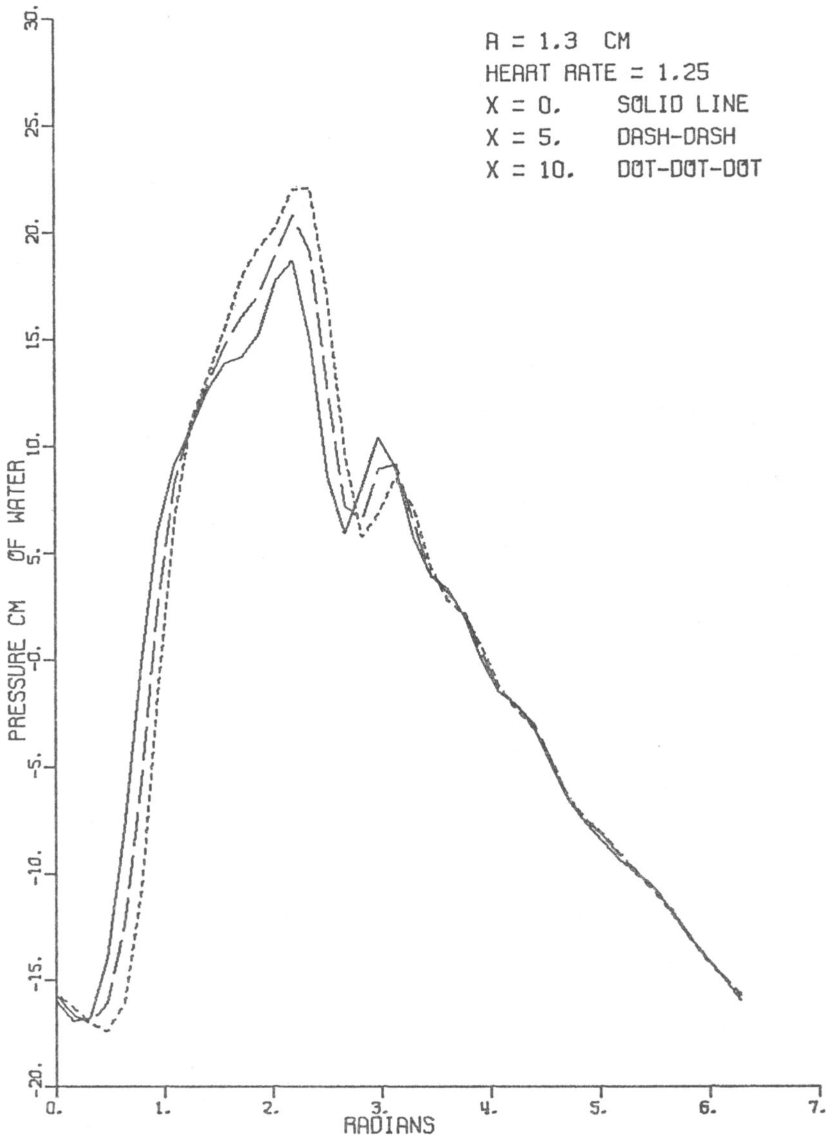

FIGURE 1 Pulse pressure vs. time (isotropic).

the velocity c_{p2} approximates the quantity $(E/\rho)^{1/2}$ which represents the velocity of a longitudinal wave through the tube wall. It is interesting to note, however, that for the orthotropic cases 2 and 3 the first wave propagates with approximately the same velocity c_{p1} as that for the isotropic case; however, the second wave propagates with markedly different velocities c_{p2}. In particular, for case 3 where all the Poisson ratios have been taken as zero, c_{p2} is not markedly different from c_{p1}. If experiments

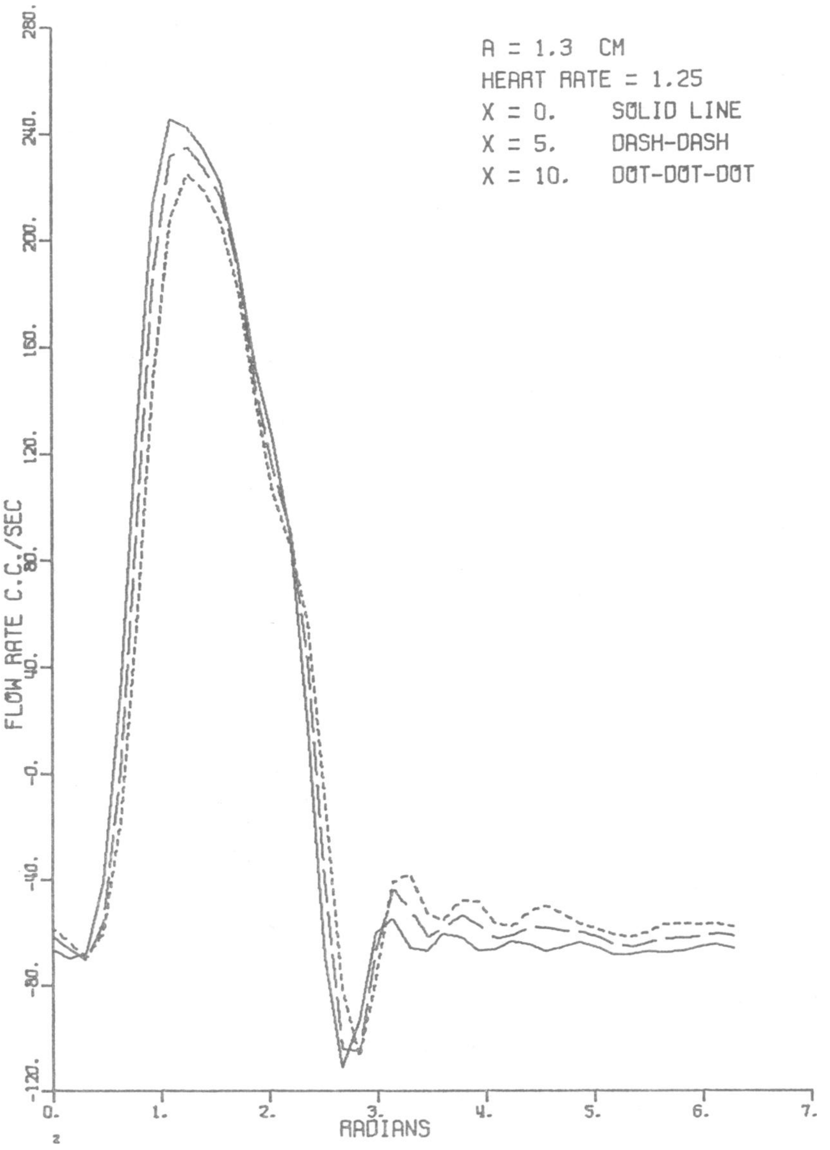

FIGURE 2 Flow rate vs. time (isotropic).

indicate the presence of such a wave, it will be necessary in future analyses to include also terms involving the velocity c_{p2}. The important conclusion from the results of Table I is that c_{p1} remains relatively constant for all three cases and depends mainly on the tangential Young's modulus E_θ. This point was brought out by Lambossy and Müller (1954); however, their conclusions were based on a static analysis.

The fluid inductance and resistance behave the same qualitatively in all three

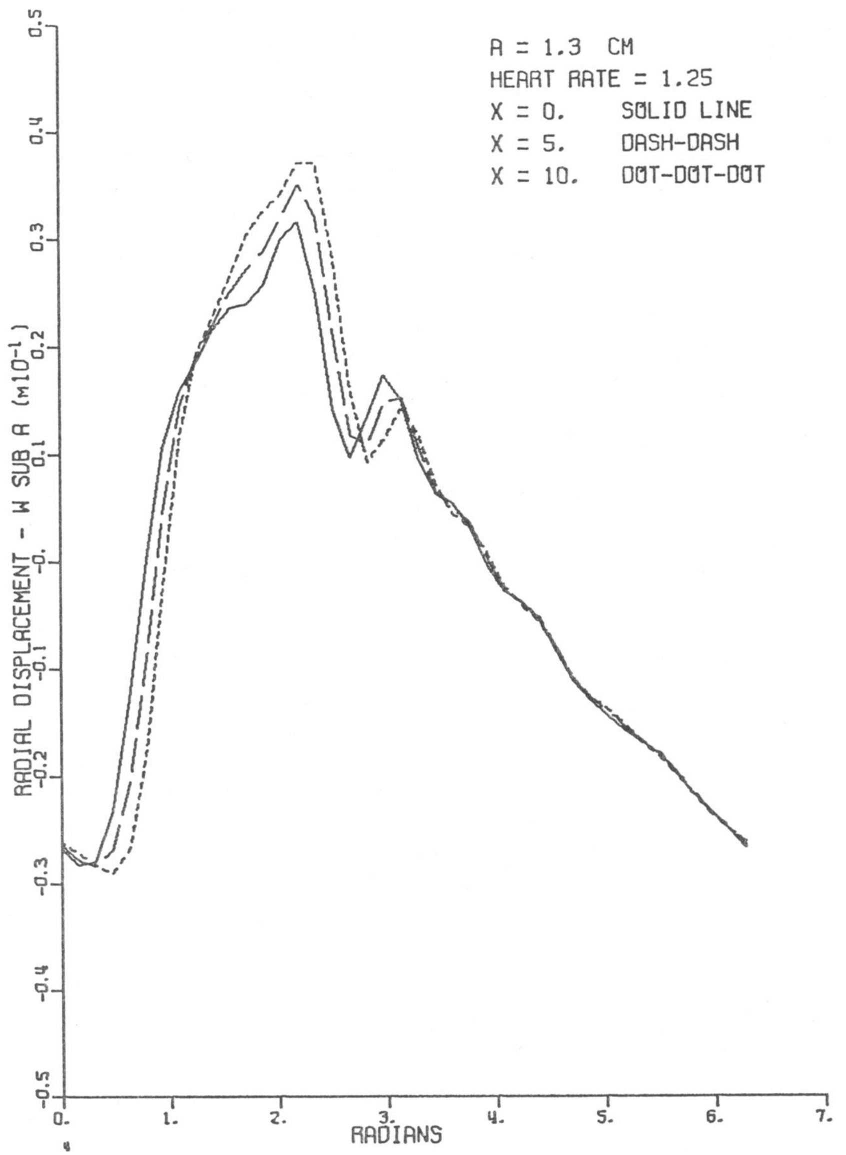

FIGURE 3 Radial displacement vs. time (isotropic).

cases. As evidenced from Table II the inductance $L_{\alpha n}$ is frequency independent for moderately large frequencies and is also independent of the orthotropicity of the tube material. However, the hydraulic resistances $R_{\alpha n}$ for the orthotropic cases are progressively larger than those for isotropy. These results may provide an alternative explanation for the findings of Fry et al. (1964) who attributed their higher values to turbulence.

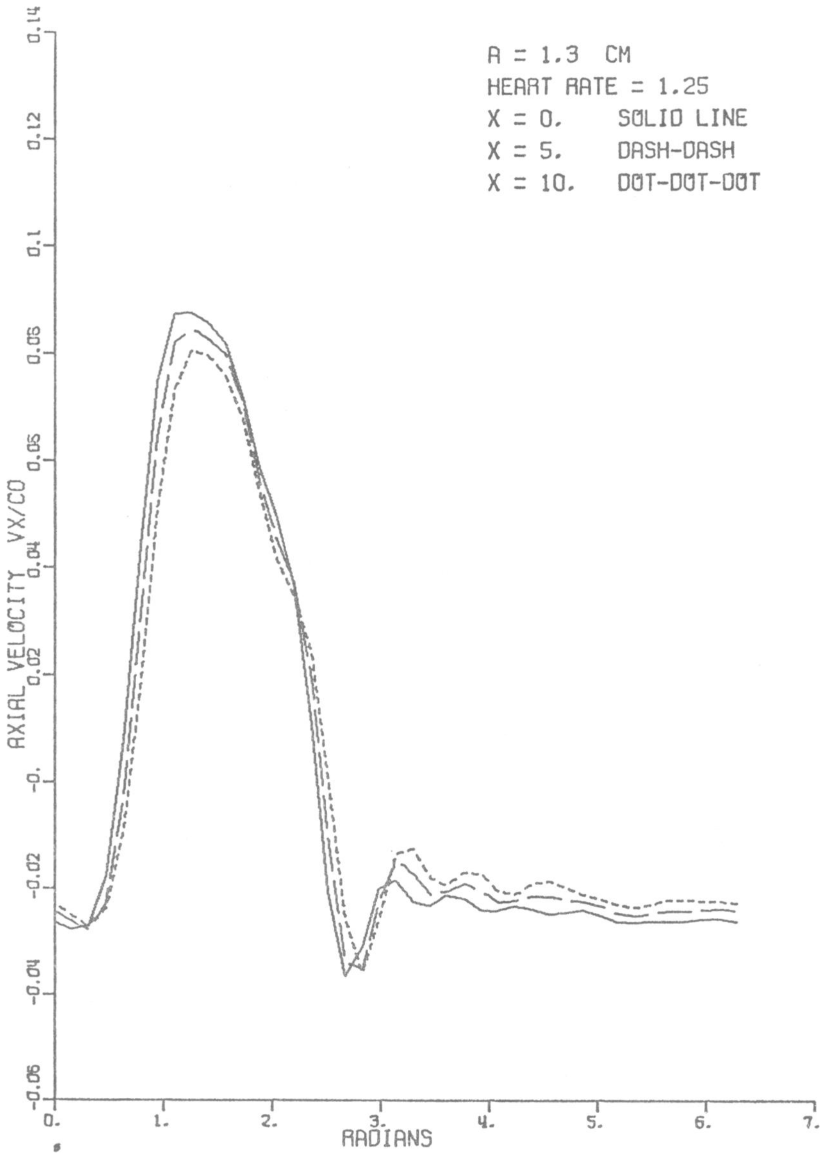

FIGURE 4 Axial velocity vs. time at $r = 0$ (isotropic).

In Figs. 1 through 7 the various physical parameters are shown plotted by means of an IBM Calcomp Plotter over one period for three different stations. For $x = 0$, the flow rate and pressure pulse curves coincide with the data obtained from the experimental studies of Patel and Fry et al. (1965). As we proceed away from the heart the pulse pressure increases in amplitude and the flow rate decreases which is found to be the case experimentally. The value $x = 10$ cm was chosen for illustrative

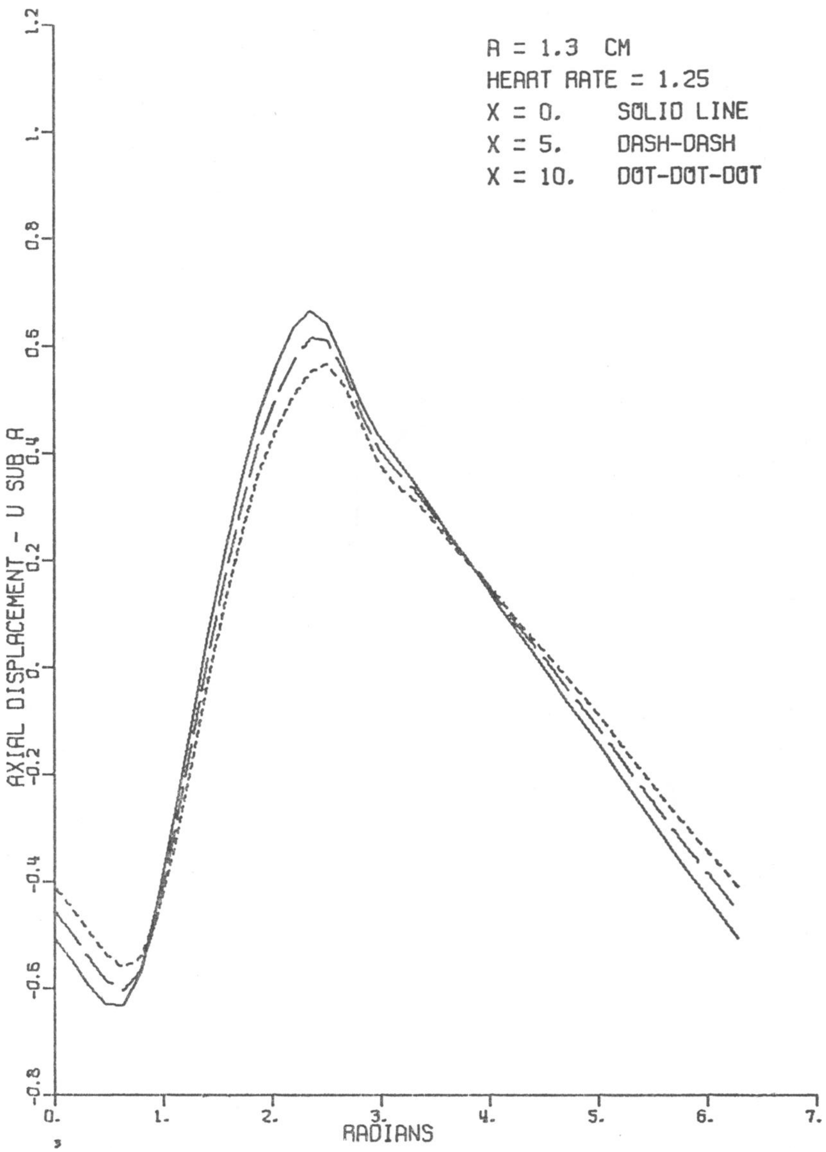

FIGURE 5 Axial displacement vs. time (isotropic).

purposes only since it is known that the ascending aorta does not extend more than 5–6 cm in length. Furthermore, it should be emphasized that the present model is valid for straight cylindrical tubes and therefore one should interpret the results for the ascending aorta with a little caution. The present analysis is more likely to be valid in the straight portion of the descending aorta before it tapers down where one would expect the flow rate to increase and the pulse pressure to decrease.

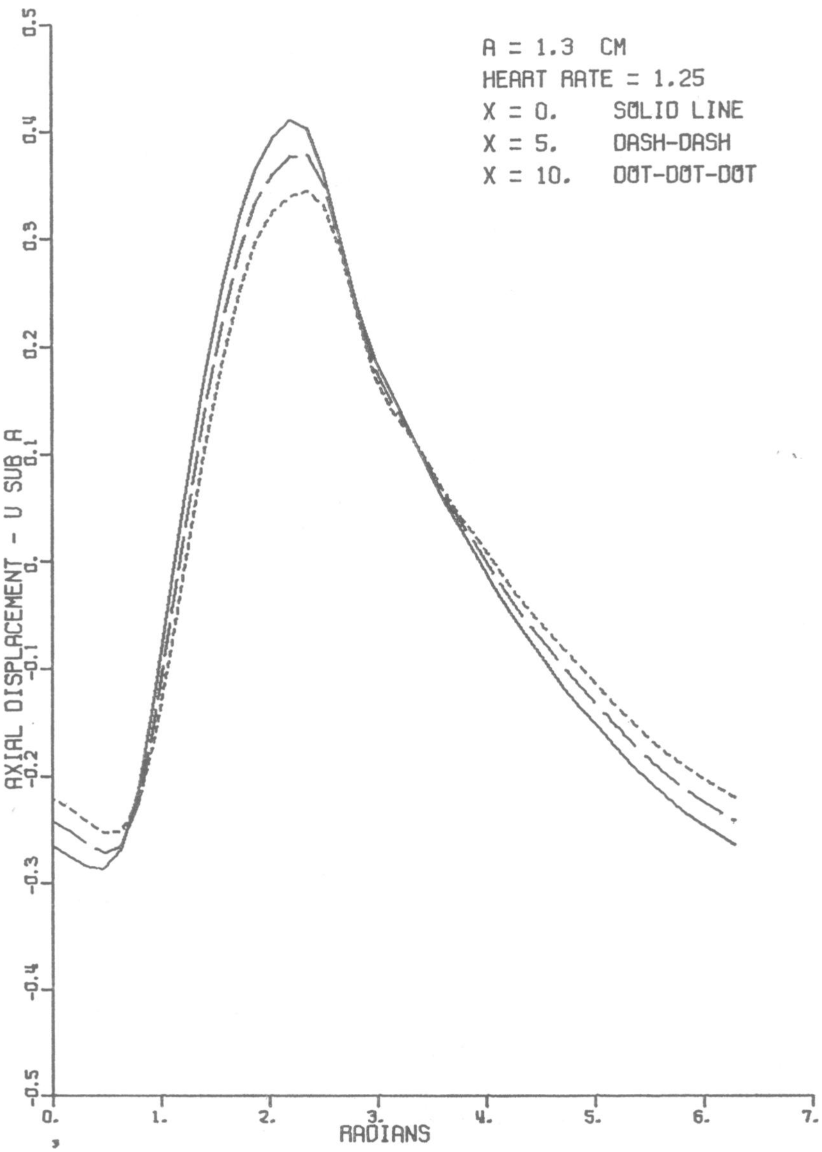

FIGURE 6 Axial displacement vs. time (orthotropic).

The plots for pressure, flow rate, radial displacement, and axial flow velocity are shown for the isotropic case only since very small variations from these curves occurred for the orthotropic cases. Fig. 3 represents a plot of the dimensionless radial

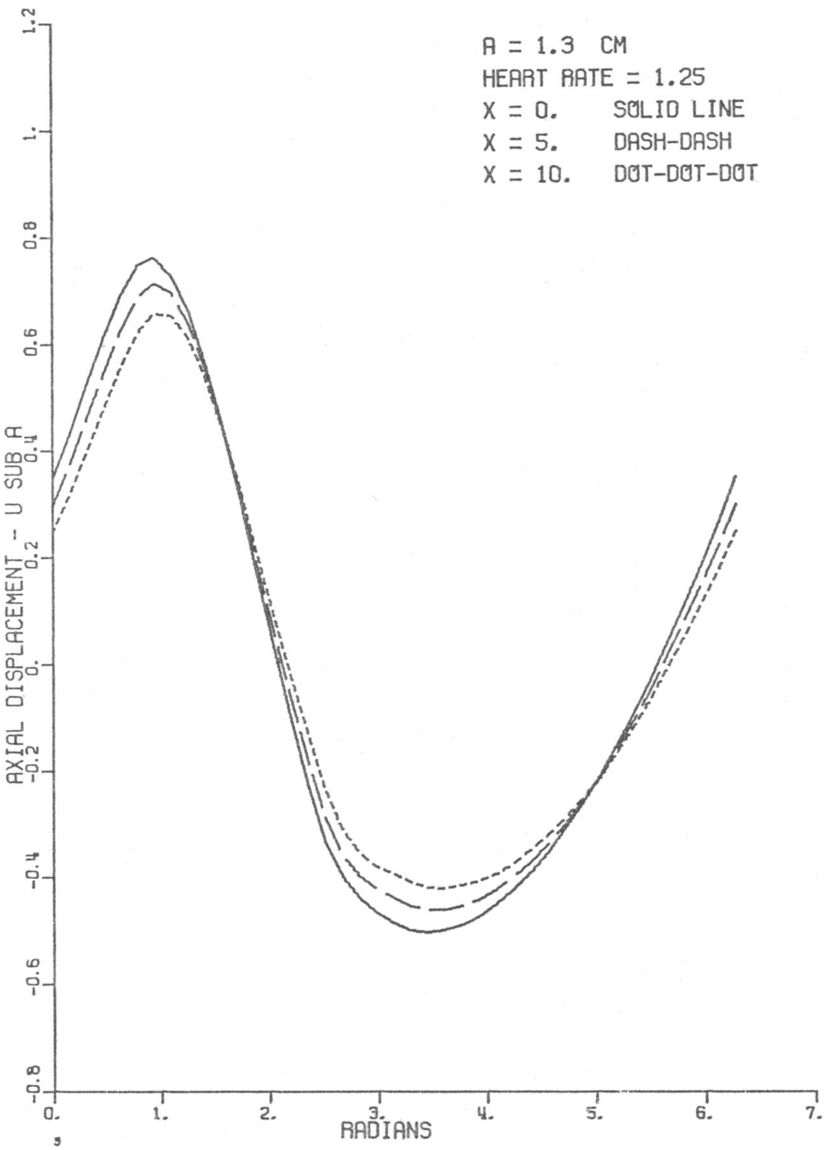

FIGURE 7 Axial displacement vs. time (orthotropic, $\nu_{ij} = 0$).

displacement on the inner surface of the tube $W_a = \frac{1}{a}\bar{u}_{za}$. The orders of magnitude of this displacement compare favorably with those obtained on the basis of a static analysis. As one would expect, the radial displacement and pressure pulse vary in a similar manner.

The axial flow velocity at the center of the tube $r = 0$ plotted in Fig. 4 is observed to be of the order of 10% of the Moens-Korteweg velocity $c_o = (E\bar{q}/2\rho_f)^{1/2}$.

Figs. 5 through 7 exhibit the marked effect of the orthotropicity of the wall material on the axial displacement $U_a = \frac{1}{a}\bar{u}_{xa}$. These results indicate a reduction in the absolute value of the displacement as one proceeds away from the heart and are consistent with the physiological situation. The longitudinal movements for the isotropic case, Fig. 5, are of the order of ⅓ of the diameter of the aorta but these are reduced to ⅕ of the diameter for the orthotropic Case 2 shown in Fig. 6. In Fig. 7 where the $\nu_{ij} = 0$ the displacements are slightly larger; however, the interesting result is that the movements are in a direction opposite to those of the previous two cases. Although these movements may still be larger than those observed experimentally, the results suggest that anisotropy must be included in future mathematical models if one is to gain a better understanding of the cardiovascular system.

The possible inclusion of initial stresses as introduced by Atabek and Lew (1966) combined with anisotropy of the wall material may provide the answer to the problem of "longitudinal tethering."

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